Introduction to countable Borel equivalence relations

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  - Classifying these points up to some notion of equivalence (e.g., conformal equivalence, isomorphism, conjugacy) means understanding the (Borel) complexity of this equivalence relation.

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- In other words, there is a Borel embedding of quotient spaces, i.e., an injection $X/E \hookrightarrow Y/F$ that lifts to a Borel map $X \to Y$.

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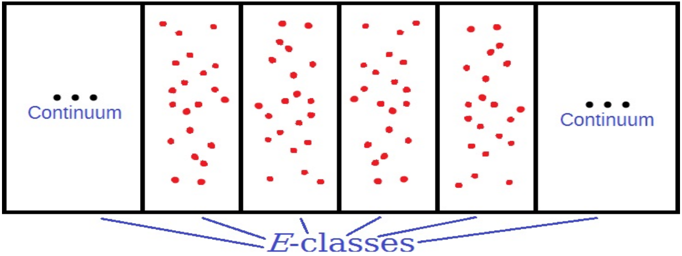
  - These connect to countable group actions and Borel graph combinatorics.

Countable Borel equivalence relations via group actions



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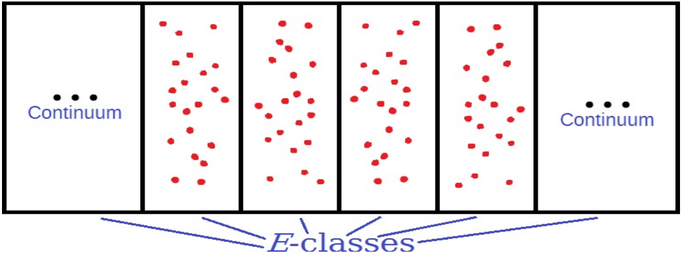

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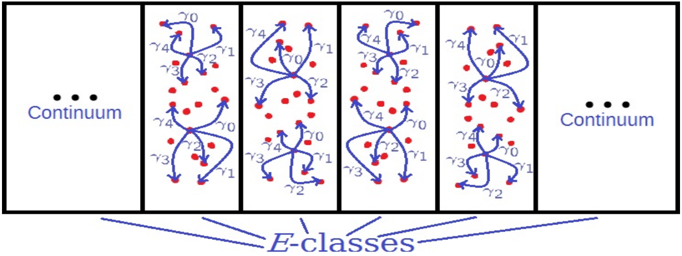

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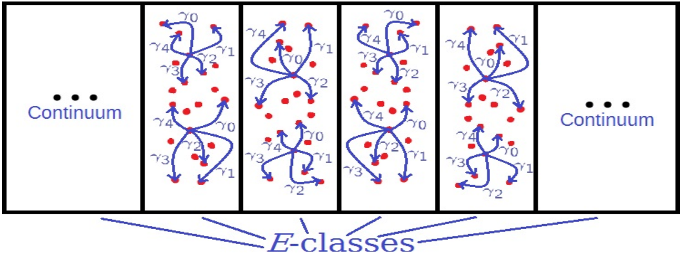

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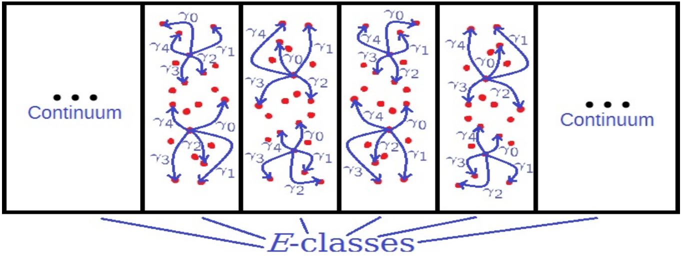

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- The abundance of Borel actions of countable groups makes the class of CBERs extremely rich.

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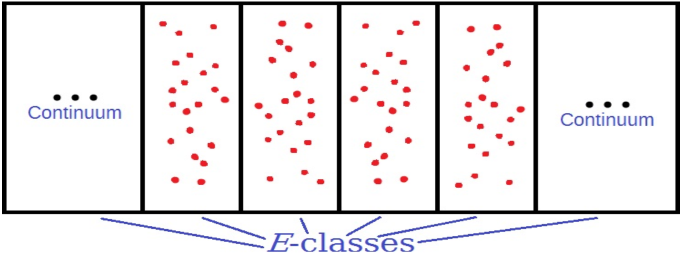

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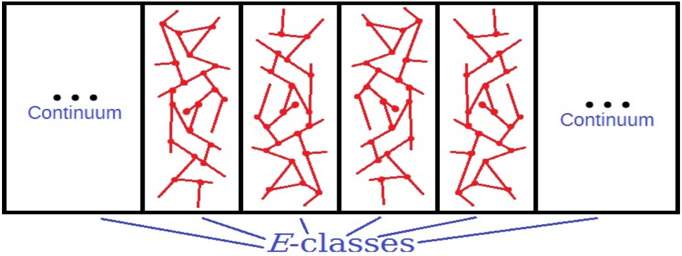

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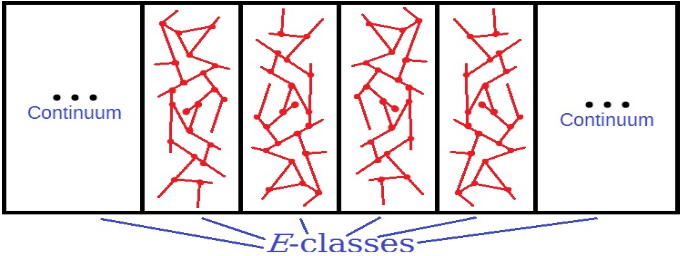

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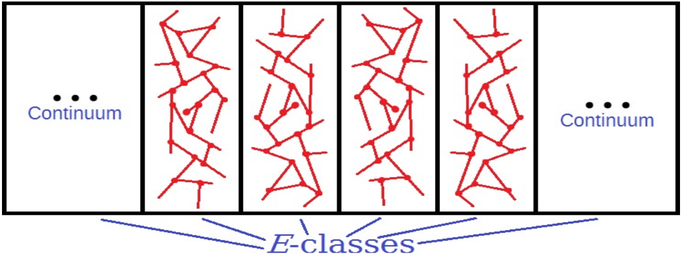

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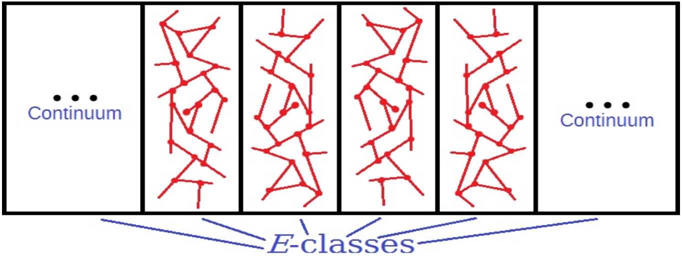

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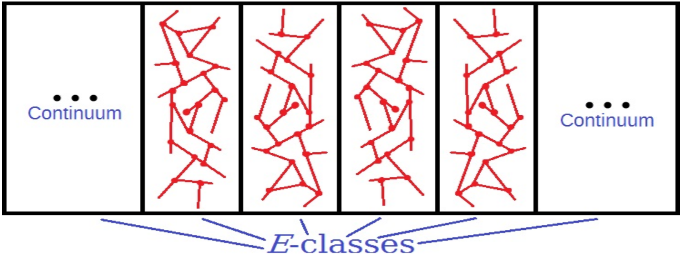

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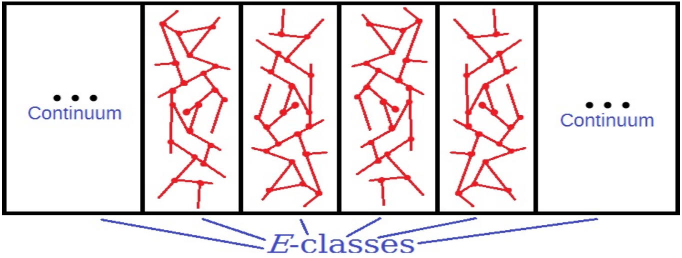

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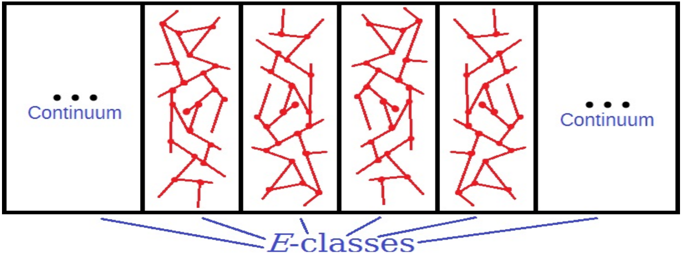

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Then $E = E_{G_S}$.

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- All these together has created extremely active two-way traffic between the study of CBERs and
  - ergodic theory
  - measured group theory
  - graph combinatorics
  - geometric group theory
  - percolation theory
  - probabilistic combinatorics
  - topological dynamics
  - von Neumann algebras

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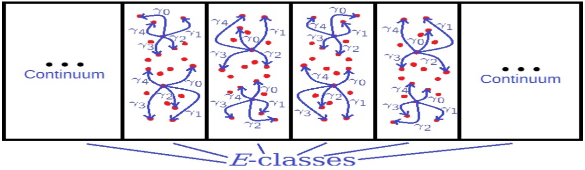

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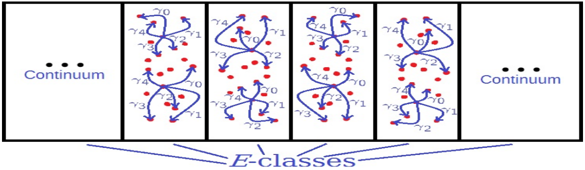

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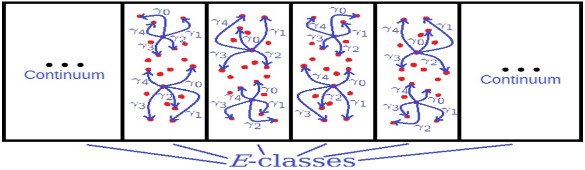

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Countable Borel equivalence relations

- Let E be an equivalence relation on X.
- E is Borel if it is a Borel subset of X^2.
- E is countable if each E-class is countable.

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Theorem (Feldman–Moore)

Each countable Borel equivalence relation E is the orbit equivalence relation of Borel action of some countable group Γ.

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Question (Rigidity for free groups—restated)

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This however, doesn't answer the original question about free groups.

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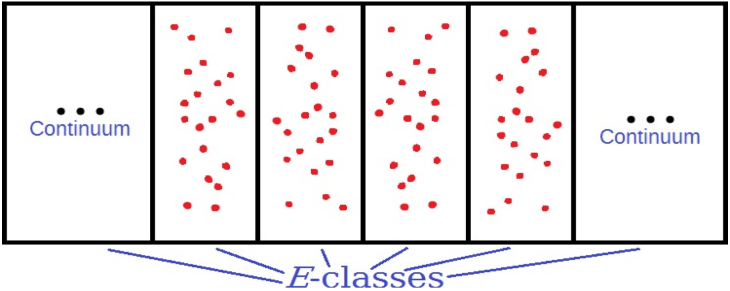

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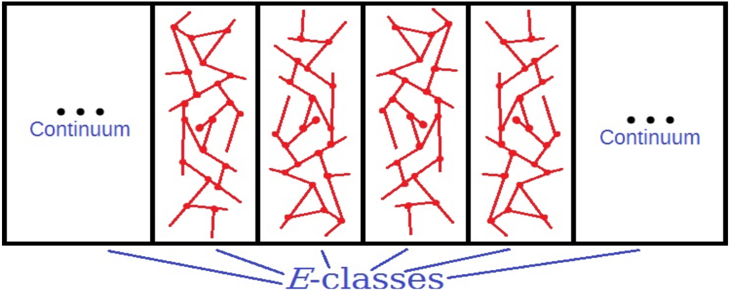
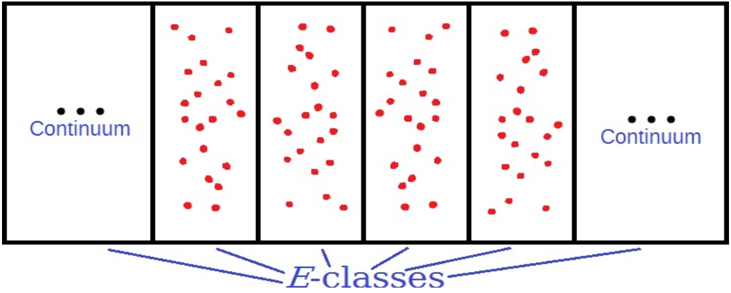


- We'll use edges between the points of X to connect each equivalence class and the minimum amount of edges will be the cost of E.

Graphings and cost

Let E be a countable Borel equivalence relation on (X, μ).



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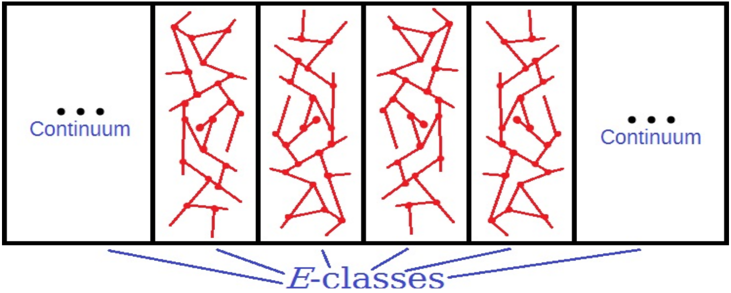

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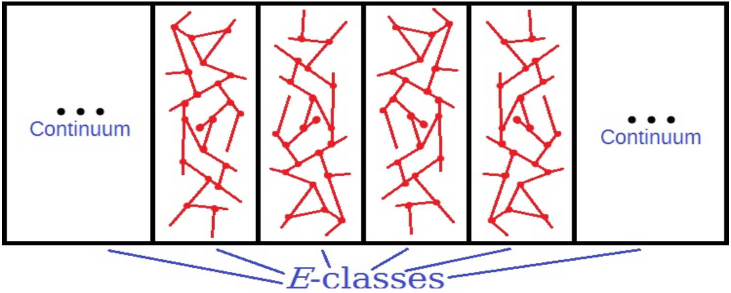

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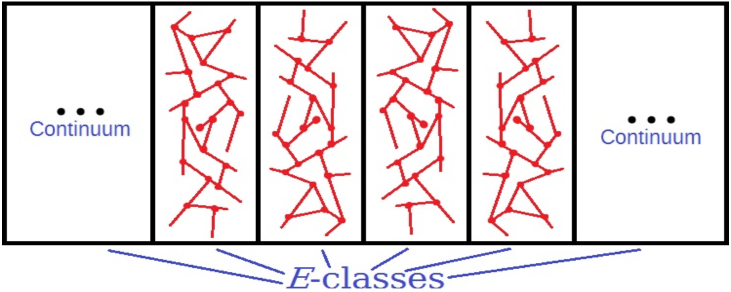

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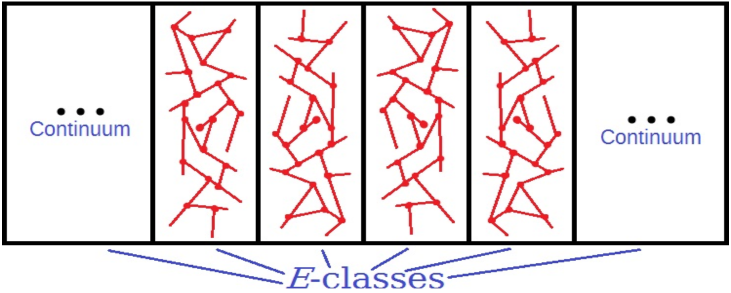

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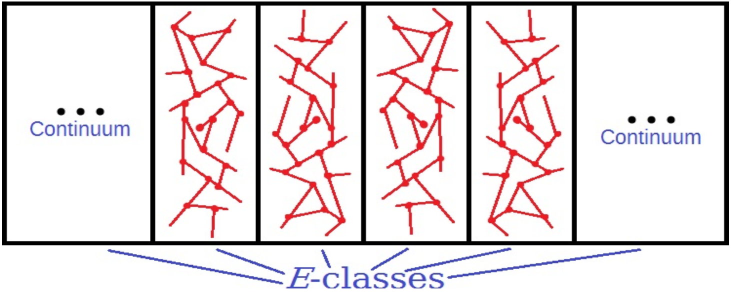

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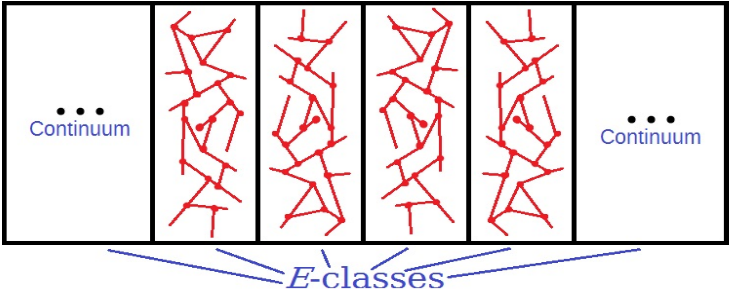

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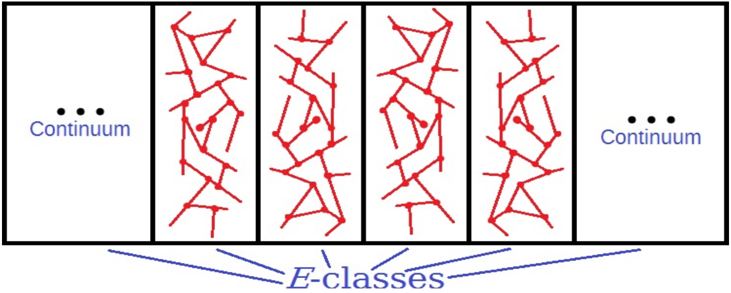

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- Minimal graphings: A graphing T of E is called a treeing if it is acyclic.

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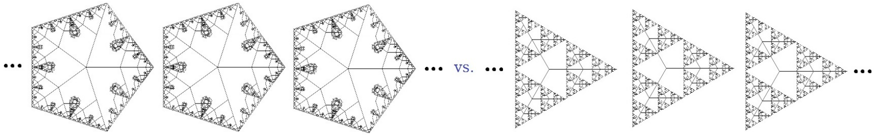

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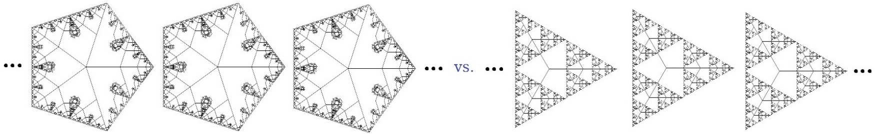

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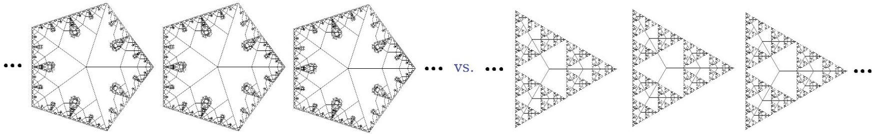

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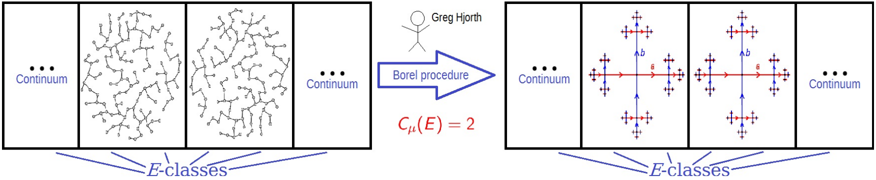

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A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

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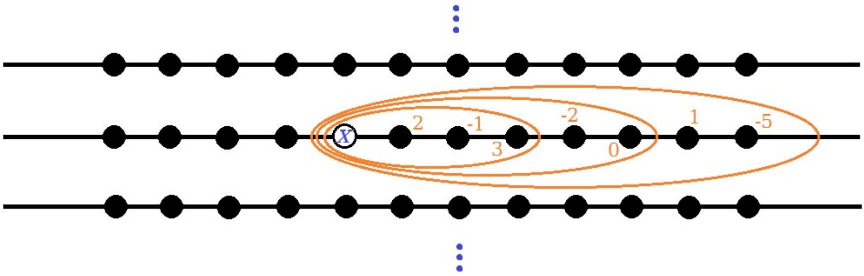

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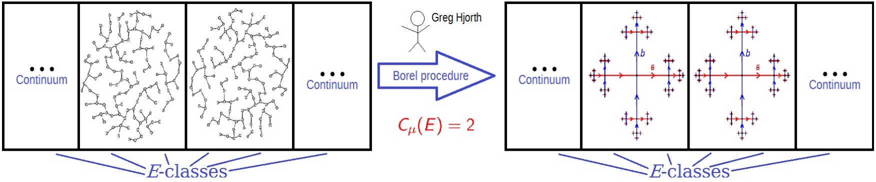

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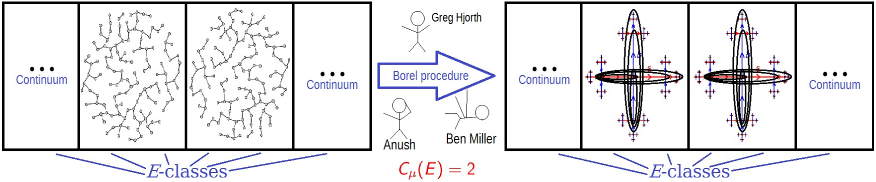

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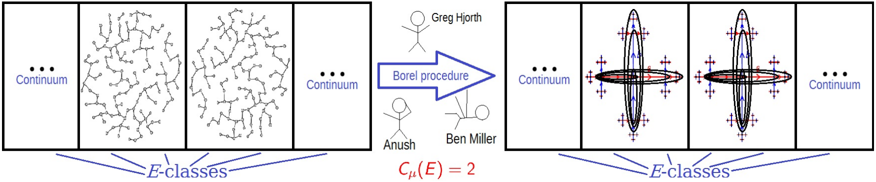

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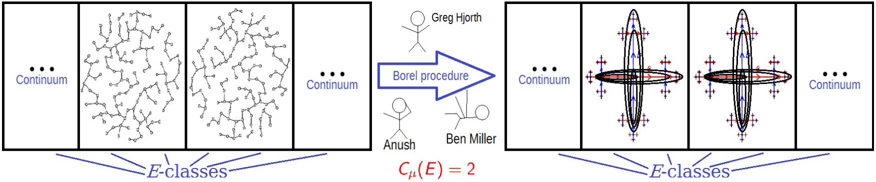

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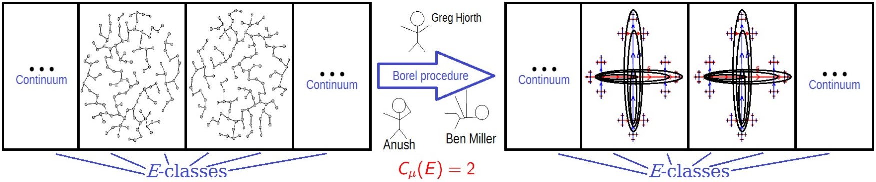

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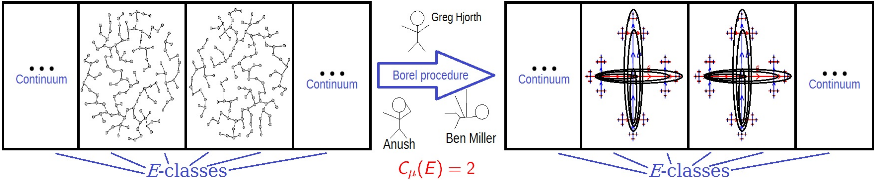

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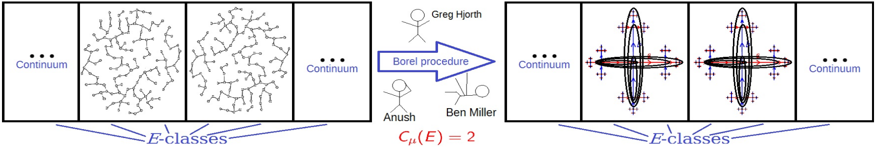
To do this, we introduce and use:

- edge sliding: a homology preserving Borel technique for modifying graphs,
- building saturated Borel partitions into finite sets,
- an easy method of exploiting nonamenability.

Edge sliding



- An edge sliding along a railway $R \subseteq X^2$ is a Borel map $\sigma : X^2 \to X^2$ that keeps the edges of R fixed and for every $e \in X^2$:

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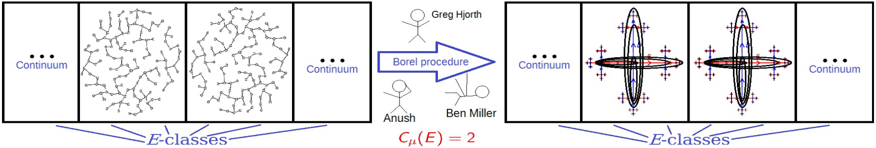

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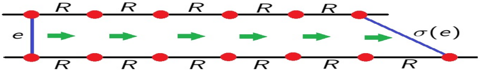

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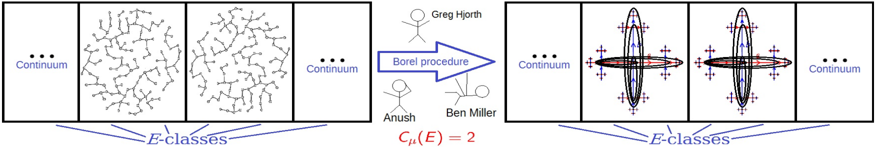

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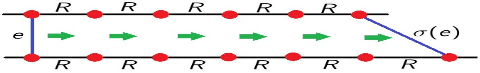

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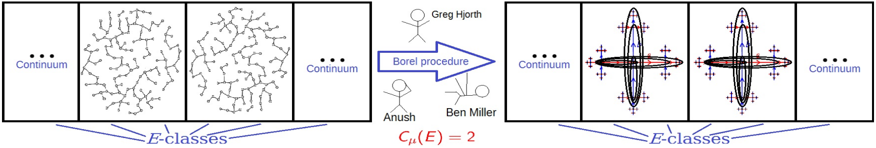

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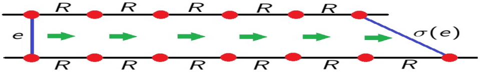

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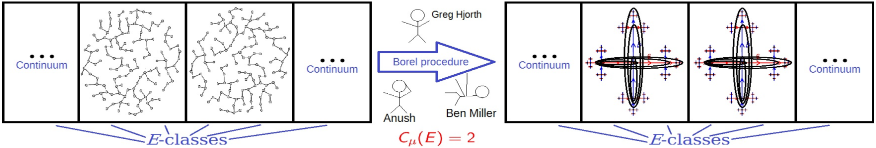

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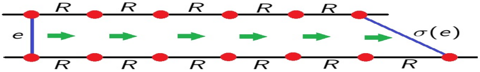

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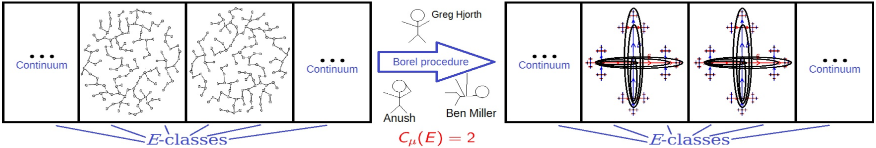

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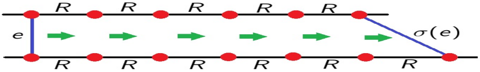

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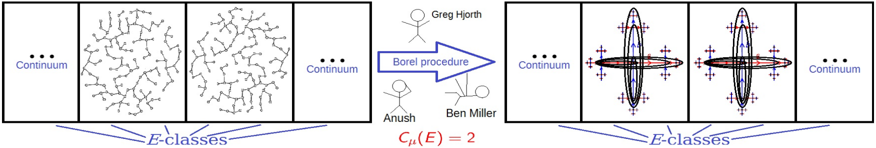

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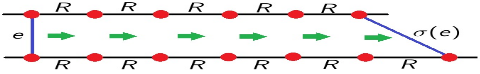

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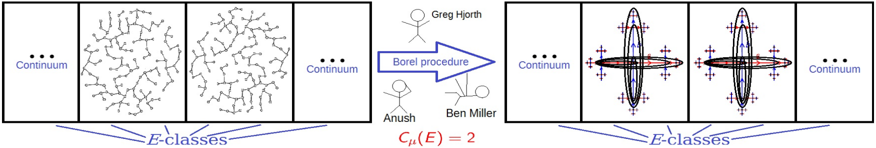

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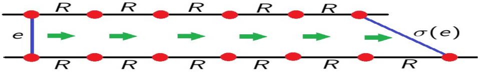

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Thanks!